CS 6505 Bloom Filters Project Report

Pavel Komarov

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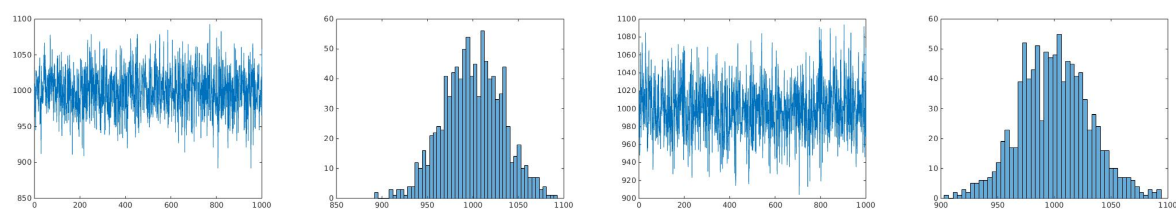
1 Introduction

Bloom filters keep an array H of n bits initially set to 0. Insertion is done by hashing an input x k times with k different hash functions and setting the corresponding k bits in H to 1. Querying is done by hashing a value y k times with the same k hash functions used for insertion and checking to see whether all k of those bits are 1. The brilliance of this approach is that both insertion and querying can be done in $O(1)$ with very, very little space, efficiencies traditional hash tables can not reach. The trouble, of course, is that there is some nonzero probability that all k bits accessed in a query were set to 1 by additions of xs that were not y. The purpose of this project is to do some analysis of that error rate.

2 Hashing Machinery: A Foray in to Probability

Early on I got stuck wondering how I could hash numbers simply for insertion and then reproduce those same hashes at query-time. Ideally, I would have k functions that all output perfectly uniform distributions given any inputs, but implementing such functions is potentially very complicated. To find a solution, I created Analysis.java and Analysis.m and tried several mathematical manipulations based on multiplication and overflow. The results were random in some cases but not for all cases. Eventually I realized a random number XOR a random number is a random number, so I could keep a vector of k random numbers and XOR with inputs to simulate a call to hash(input).

To prove the validity of this approach, I ran simulations. For two simulated hashes, **Figure 1** depicts the contents of $n = 1000$ "bins" filled with $m = 1000000$ "balls" ((a) and (c)) as well as histograms of the number of bins that contain $b \pm q$ balls ((b) and (d)). For all simulated hashes I have tried (dozens), the results look like this white noise.



(a) (b) (c) (d)

Figure 1: Analysis of simulated hash function demonstrating that input XOR kth random number is valid.

To make further sense of the above, recall that the probability a given bin i in H holds b balls is given by Bernoulli trials as:

$$Pr[H_i = b] = \binom{m}{b} \left(\frac{1}{n}\right)^b \left(1 - \frac{1}{n}\right)^{m-b}$$

By the central limit theorem, as the number of trials goes to infinity, the distribution of these new random variables is given by a Gaussian iff the underlying trials are independent, which is as I observe. So, since these hash-events are independent, and since the mean of the output is centered around $E[balls\ per\ bin] = \frac{m}{n}$, this method of simulating hashes does exactly what a real hash function would do.

3 Theory: The Math

First, a derivation of the theoretical error rate for Bloom Filters. This math is important to fully understand the results given hereafter.

$$Pr(false\,positive|input\ y) = Pr(H[h_1(y)] = 1 \wedge H[h_2(y)] = 1 \wedge ...H[h_k(y)] = 1),\ where\ h_x\ is\ a\ hash\ function$$

$$hashes\ are\ indep. \implies above = Pr(H[h_1(y)] = 1) \wedge Pr(H[h_2(y)] = 1) \wedge ...Pr(H[h_k(y)] = 1) = Pr(H[bit] = 1)^k$$

$$Pr(H[bit] = 1) = 1 - Pr(H[bit] = 0),\ Pr(H[bit] = 0|1\ trial) = (1 - 1/n)$$

$$|insertions| \cdot |hashfunctions| = mk\ trials \rightarrow Pr(H[bit] = 0|mk\ trials) = (1 - 1/n)^{mk}$$

$$\implies Pr(false\ positive|y,\ m\ insertions) = (1 - (1 - 1/n)^{mk}))^k$$

$$But:\ (1 - 1/n) \approx e^{-1/n}\ for\ large\ n,\ and\ \lim_{n\to\infty}(1 - 1/n) = \lim_{n\to\infty} e^{-1/n} = 1,\ so:$$

$$Pr(false\ positive|y,\ m\ insertions) \approx (1 - e^{-mk/n})^k\ (= as\ n \rightarrow \infty)$$

4 Simulation and Results

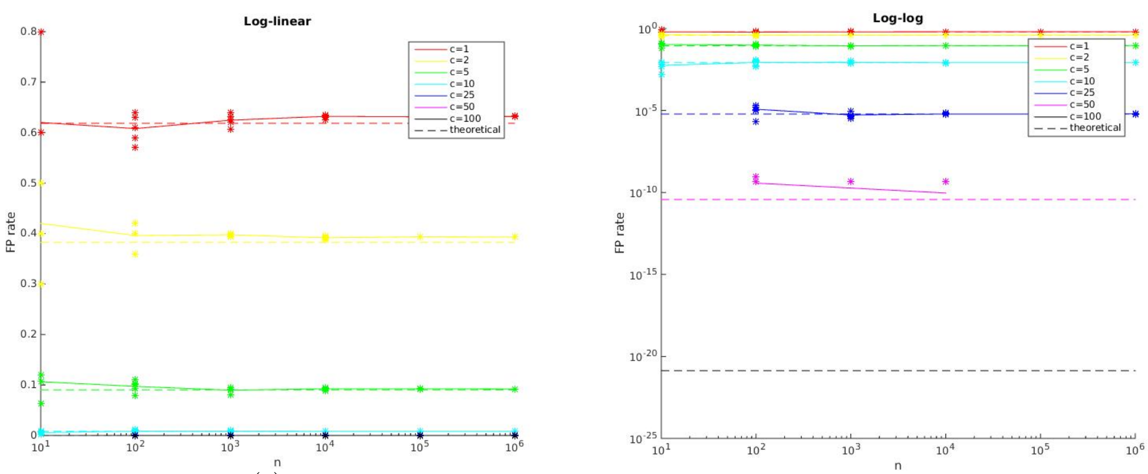
My BloomFilter.java code implements the insert and query functionality described in the intro backed by an array of $\lceil \frac{n}{32} \rceil$ ints and utilizing the simulated hashes described in **Section 2**. I use $k = \frac{n}{m}ln(2)$ (or the nearest integer) hash functions, the number which minimizes false positives, as found from the critical point of $(1 - e^{-mk/n})^k$. See my code for all the details; it's fairly straight-forward.

The simulation itself works by iterating over $n \in \{10, 10^2, 10^3, 10^4, 10^5, 10^6\}$ and $c = \frac{n}{m} \in \{1, 2, 5, 10, 25, 100\}$ (which all divide n evenly). On each pass I set up the Bloom Filter anew for that (n, c), insert m random numbers, query over [0,max int-1] (my chosen universe) to find all "true" responses, calculate the false-positive rate with

$$rate_{FP} = \frac{|FP|}{|FP| + |TN|} = \frac{|P| - |TP|}{|Universe| - |TP|},$$

and finally write results to a file. I force a repeat trial for each (n, c) pair five times so I can see variance.

After a couple of hours for run-time, I processed the results in Matlab with BloomFilter.m to generate the plots shown in **Figure 2**. Since $rate_{FP} \rightarrow 0$ as $c \rightarrow \infty$, I have shown the data on a log-log scale in addition to log-linear.



(a) (b)

Figure 2: Bloom Filter false positive rates for multiple values of c as n increases.

One should notice immediately that increasing c does wonders for the error rate. At c = 50, I only observe a false positive on a small fraction of my trials, and at c = 100, I observe none. The reason for this should be clear: My universe is $2^{31} - 1$ elements large, and the theoretical error rate for this c is $\frac{1}{2}^{100ln(2)}$. The product of these numbers, the probability I see a false positive on any given run, is $2.92 \cdot 10^{-12}$!

The other important result from this analysis is less obvious but just as important. Notice there is more variation between datapoints at small n, but as n grows this variance shrinks, and trials fit more tightly to the theoretical prediction. The reason for this is in the math: Remember the functions $(1 - 1/n)$ and $e^{-1/n}$ are better approximations of each other as $n \rightarrow \infty$.

5 Conclusion

This class is the first place I have ever heard of Bloom Filters. Considering how fast they are, how little space they take, and how low their error rates can be, I will keep them in mind for applications where a very few false-positives is okay.

I am including all of the files I used to generate this report in my submission, probably more than you want. Focus on BloomFilter.java if you only have time for one.